PRIME NON-COMMUTATIVE JB^*-ALGEBRAS

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Abstract

We prove that, if A is a prime non-commutative JB^*-algebra, and if A is neither quadratic nor commutative, then there exist a prime C^*-algebra B and a real number λ with $\frac{1}{2} < \lambda \leq 1$ such that $A = B$ as involutive Banach spaces, and the product of A is related to that of B (denoted by $\circ$, say) by means of the equality $xy = \lambda x \circ y + (1 - \lambda) y \circ x$.

0.- Introduction

Non-commutative JB^*-algebras are the non-associative counterparts of C^*-algebras. They arise in Functional Analysis by the hand of the general non-associative extension of the Vidav-Palmer theorem. Indeed, norm-unital complete normed non-associative complex algebras subjected to the geometric Vidav condition characterizing C^*-algebras in the associative setting [2; Theorem 38.14] are nothing but unital non-commutative JB^*-algebras [10; Theorem 12]. The classical structure theory for non-commutative JB^*-algebras consists of a precise classification of certain prime non-commutative JB^*-algebras (the so-called " non-commutative JBW^*-factors ") and the fact that every non-commutative JB^*-algebra has a faithful family of factor representations (see [1], [3], [8], and [9]).

In this paper we obtain a classification of all prime non-commutative JB^*-algebras, which extends that of non-commutative JBW^*-factors. Precisely, we prove that, if A is a prime non-commutative JB^*-algebra, and if A is neither quadratic nor commutative, then there exist a prime C^*-algebra B and a real number λ with $\frac{1}{2} < \lambda \leq 1$ such that $A = B$ as involutive Banach spaces, and the product of A is related to that of B (denoted by $\circ$, say) by means of the equality $xy = \lambda x \circ y + (1 - \lambda) y \circ x$. We note that prime non-commutative JB^*-algebras which are either quadratic or commutative are well-understood (see [9, Section 3] and the Zel'manov-type prime theorem for JB^*-algebras [5; Theorem 2.3], respectively). We also note that our result becomes a natural analytical variant of the classification theorem for prime nondegenerate non-commutative Jordan algebras, proved by W. G. Skosyrskii [12].

1.- The results

Following [11; p. 141], we define *non-commutative Jordan* algebras as those algebras A satisfying the *Jordan identity* $(xy)x^2 = x(yx^2)$ and the *flexibility condition* $(xy)x = x(yx)$. For an element x in a non-commutative Jordan algebra A, we denote by U_x the mapping $y \to x(xy+yx)-x^2y$ from A to A. By a *non-commutative JB^*-algebra* we mean a complete normed non-commutative Jordan complex algebra (say A) with conjugate-linear algebra involution $*$ satisfying $\| U_x(x^*) \| = \| x \|^3$ for every x in A. When a non-commutative JB^*-algebra A is actually commutative, we simply say that A is a JB^*-algebra. We note that C^*-algebras are precisely those non-commutative JB^*-algebras which are associative. Since non-commutative Jordan algebras are power-associative (i.e., all their one-generated subalgebras are associative) [11; p. 141], it follows that the closed subalgebra of a non-commutative JB^*-algebra generated by any of its self-adjoint elements is a commutative C^*-algebra.

Clearly, the ℓ_∞-sum of every family of non-commutative JB^*-algebras is a non-commutative JB^*-algebra in a natural manner. Then, the fact that $*$-homomorphisms between non-commutative JB^*-algebras are contractive (in fact isometric whenever they are injective) [8; Proposition 2.1] leads to the following folklore result.

LEMMA 1- *Let A be a non-commutative JB^*-algebra, I a non-empty set, and, for each i in I, let φ_i be a $*$-homomorphism from A into a non-commutative JB^*-algebra A_i. If $\cap_{i\in I} Ker(\varphi_i) = 0$, then we have*

$$\| x \| = \sup\{\| \varphi_i(x) \| \quad : i \in I\}$$

for every x in A.

To continue our argument, we need to invoke some techniques of Banach ultraproducts [7]. Let I be a non-empty set, $\mathcal{U}$ an ultrafilter on I, and $\{X_i\}_{i\in I}$ a family of Banach spaces. We may consider the Banach space $(\oplus_{i\in I} X_i)_\infty$, ℓ_∞-sum of this family, and the closed subspace $N_\mathcal{U}$ of $(\oplus_{i\in I} X_i)_\infty$ given by

$$N_\mathcal{U} := \{\{x_i\} \in (\oplus_{i\in I} X_i)_\infty : \lim_\mathcal{U} \| x_i \| = 0\}.$$

The *(Banach) ultraproduct* $(X_i)_\mathcal{U}$ of the family $\{X_i\}_{i\in I}$ relative to the ultrafilter $\mathcal{U}$ is defined as the quotient Banach space $(\oplus_{i\in I} X_i)_\infty / N_\mathcal{U}$. If we denote by (x_i) the element in $(X_i)_\mathcal{U}$ containing a given family $\{x_i\} \in (\oplus_{i\in I} X_i)_\infty$, then it is easy to verify that $\| (x_i) \| = \lim_\mathcal{U} \| x_i \|$. We note that, if, for every i in I, X_i is a non-commutative JB^*-algebra, then $N_\mathcal{U}$ is a closed (two-sided) ideal of the non-commutative JB^*-algebra $(\oplus_{i\in I} X_i)_\infty$, and therefore, by [8; Corollary 1.11], $(X_i)_\mathcal{U}$ is a non-commutative JB^*-algebra in a natural way.

Let A be an algebra. A is said to be *prime* if $A \neq 0$ and, whenever P, Q are ideals of A with $PQ = 0$, we have either $P = 0$ or $Q = 0$. If A is prime, and if P, Q are ideals of A satisfying $P \cap Q = 0$, then, clearly, either $P = 0$

or $Q = 0$. Assume that A is prime, and let $\{P_i\}_{i\in I}$ be a family of ideals of A such that $\cap_{i\in I} P_i = 0$. For x in $A\backslash\{0\}$, put $I_x := \{i \in I : x \notin P_i\}$. Then $\mathcal{B} := \{I_x : x \in A\backslash\{0\}\}$ is a filter basis on I. Indeed, it follows from the assumption $\cap_{i\in I} P_i = 0$ that I_x is non-empty for every x in $A\backslash\{0\}$, whereas, for x, y in $A\backslash\{0\}$, we have $0 \neq x \in \cap_{i\in I\backslash I_x} P_i$ and $0 \neq y \in \cap_{i\in I\backslash I_y} P_i$, hence, by the primeness of A, there exists $0 \neq z \in \cap_{i\in I\backslash(I_x\cap I_y)} P_i$, so that $I_z \subseteq I_x \cap I_y$.

PROPOSITION 2.- *Let A be a prime non-commutative JB^*-algebra, I a non-empty set, and, for each i in I, let φ_i be a $*$-homomorphism from A into a non-commutative JB^*-algebra A_i. Assume that $\cap_{i\in I} Ker(\varphi_i) = 0$. Then there exists an ultrafilter $\mathcal{U}$ on I such that the $*$-homomorphism $\varphi : x \to (\varphi_i(x))$ from A to $(A_i)_{\mathcal{U}}$ is injective.*

Proof.- For i in I, put $P_i := Ker(\varphi_i)$, and let $\mathcal{B}$ be the filter basis on I associated to the family $\{P_i\}_{i\in I}$ as in the previous comment. Take an ultrafilter $\mathcal{U}$ on I containing $\mathcal{B}$. Suppose that the mapping $\varphi : x \to (\varphi_i(x))$ from A to $(A_i)_{\mathcal{U}}$ is not injective. Then there exists x in A satisfying $\| x \| = 1$ and $\lim_{\mathcal{U}} \varphi_i(x) = 0$. Therefore $J := \{i \in I : \ \| \varphi_i(x) \| < \frac{1}{2}\}$ is an element of $\mathcal{U}$. But, from the fact that $\| x \| = 1$, the definition of J, and Lemma 1, we obtain $\cap_{i\in J} P_i \neq 0$. Then, taking a non-zero element y in $\cap_{i\in J} P_i$, we have $J \cap I_y = \emptyset$. Since J and I_y are elements of $\mathcal{U}$, this is a contradiction. ■

Let $\mathbb{F}$ be a field containing more than two elements. Following [11, pp. 49-50], an algebra A over $\mathbb{F}$ is called *quadratic* (over $\mathbb{F}$) if it has a unit $\mathbf{1}$, $A \neq \mathbb{F}\mathbf{1}$, and, for each x in A, there are elements $t(x)$ and $n(x)$ in $\mathbb{F}$ such that $x^2 - t(x)x + n(x)\mathbf{1} = 0$. If A is a quadratic algebra over $\mathbb{F}$, then, for x in $A \setminus \mathbb{F}\mathbf{1}$, the scalars $t(x)$ and $n(x)$ are uniquely determined, so that, choosing $t(\alpha\mathbf{1}) := 2\alpha$ and $n(\alpha\mathbf{1}) := \alpha^2$ $(\alpha \in \mathbb{F})$, we obtain mappings t and n (called the *trace form* and the *algebraic norm*, respectively) from A to $\mathbb{F}$, which are linear and quadratic, respectively (see again [11; pp. 49-50]).

LEMMA 3.- *Let A be a quadratic non-commutative JB^*-algebra. Then we have $| t(x) | \leq 2 \| x \|$ and $| n(x) | \leq \| x \|^2$ for all x in A. Moreover, if B is a $*$-invariant subalgebra of A with $\dim(B) \geq 2$, then the unit of A lies in B, and therefore B is a quadratic algebra.*

Proof.- For x in A, the spectrum of x relative to the (associative and finite-dimensional) subalgebra of A generated by x consists of the roots (say λ_1, λ_2) of the complex polynomial $\lambda^2 - t(x)\lambda + n(x)$, so $t(x) = \lambda_1 + \lambda_2$ and $n(x) = \lambda_1\lambda_2$, and so $| t(x) | \leq 2 \| x \|$ and $| n(x) | \leq \| x \|^2$. Let B be a non-zero *-invariant subalgebra of A with $\mathbf{1} \notin B$. Put $A_{sa} := \{x \in A : x^* = x\}$ and $B_{sa} := B \cap A_{sa}$. Then A_{sa}, endowed with the product $(x, y) \to \frac{1}{2}(xy + yx)$, becomes a quadratic algebra over $\mathbb{R}$. Since $\mathbf{1} \notin B$, for x in B_{sa} we must have $n(x) = 0$, so $x^2 = t(x)x$, and so

$$\| x \|^2 = \| x^2 \| = | t(x) | \| x \| .$$

Now the restriction of t to B_{sa} is a linear functional on B with zero kernel, and hence the real vector space B_{sa} is one-dimensional. Since B is *-invariant, we deduce that B is one-dimensional (over $\mathbb{C}$). ■

To conclude the proof of our main result, we need some background on factor representations of non-commutative JB^*-algebras. First of all, we note that, if A is a non-commutative JB^*-algebra, and if λ is a real number with $0 \leq \lambda \leq 1$, then the involutive Banach space of A, endowed with the product

$$(x, y) \rightarrow \lambda xy + (1 - \lambda)yx \ ,$$

becomes a non-commutative JB^*-algebra (which will be denoted by $A^{(\lambda)}$). By a *non-commutative JBW^*-algebra* we mean a non-commutative JB^*-algebra which is a dual Banach space. Prime non-commutative JBW^*-algebras are called *non-commutative JBW^*-factors*. A non-commutative JBW^*-factor is said to be of *Type I* if the closed unit ball of its predual has extreme points (compare [9; Theorem 1.11]). If A is a non-commutative JBW^*-factor of Type I, and if A is neither quadratic nor commutative, then there exist a complex Hilbert space H and a real number λ with $\frac{1}{2} < \lambda \leq 1$ such that, denoting by B the C^*-algebra of all bounded linear operators on H, we have $A = B^{(\lambda)}$ [9; Theorem 2.7]. A *factor representation* of a non-commutative JB^*-algebra A is a w^*-dense range $*$-homomorphism from A into some non-commutative JBW^*-factor. Finally, let us recall that every nonzero non-commutative JB^*-algebra has a faithful family of Type I factor representations [9; Corollary 1.13].

THEOREM 4.- *Let A be a prime non-commutative JB^*-algebra. Then one of the following assertions hold for A:*

1. *A is commutative.*
2. *A is quadratic.*
3. *There exist a prime C^*-algebra B and a real number λ with $\frac{1}{2} < \lambda \leq 1$ such that $A = B^{(\lambda)}$.*

Proof.- Take a faithful family of Type I factor representations of A (say $\{\varphi_i : A \rightarrow A_i\}_{i \in I}$). Define

$$I_1 := \{i \in I \ : \ A_i \text{ is commutative}\} \ ,$$

$$I_2 := \{i \in I \ : \ A_i \text{ is quadratic}\} \ ,$$

$$I_3 := \{i \in I \ : \ A_i = B_i^{(\lambda_i)} \text{ for some } C^* - \text{algebra } B_i \text{ and some } \frac{1}{2} < \lambda_i \leq 1\} \ ,$$

and, for $n = 1, 2, 3$, write $Q_n := \cap_{i \in I_n} Ker(\varphi_i)$. Since $\cap_{n=1}^{3} Q_n = \cap_{i \in I} Ker(\varphi_i) = 0$, the primeness of A gives the existence of $m = 1, 2, 3$ such that $Q_m = 0$. Therefore, replacing I_m with I, there is no loss of generality in supposing that one of the conditions which follow is fulfilled:

1. A_i is commutative for all i in I.

2. A_i is quadratic for all i in I.

3. For each i in I, there exist a C^*-algebra B_i and some $\frac{1}{2} < \lambda_i \leq 1$ such that $A_i = B_i^{(\lambda_i)}$.

Assume that Condition 1 is satisfied. Then, clearly, A is commutative.

Now, assume that Condition 2 is fulfilled. Then, by Proposition 2, there exists an ultrafilter $\mathcal{U}$ on I such that the $*$-homomorphism $\varphi : x \to (\varphi_i(x))$ from A to $(A_i)_{\mathcal{U}}$ is injective. Note that, since, for i in I, A_i has a unit $\mathbf{1}_i$, $(A_i)_{\mathcal{U}}$ has also a unit $\mathbf{1} = (\mathbf{1}_i)$. Note also that, since quadratic algebras have dimension ≥ 2, and A has a quadratic factor representation, we have $\dim(A) \geq 2$, so $\dim(\varphi(A)) \geq 2$ (because φ is injective), and so $(A_i)_{\mathcal{U}} \neq \mathbb{C}\mathbf{1}$. For i in I, let t_i and n_i be the trace form and the algebraic norm, respectively, on the quadratic non-commutative JB^*-algebra A_i. By the first assertion in Lemma 3, for (x_i) in $(A_i)_{\mathcal{U}}$, $\{t_i(x_i)\}_{i\in I}$ and $\{n_i(x_i)\}_{i\in I}$ are bounded families of complex numbers, and therefore $t : (x_i) \to \lim_{\mathcal{U}} t_i(x_i)$ and $n : (x_i) \to \lim_{\mathcal{U}} t_i(x_i)$ become well-defined mappings from $(A_i)_{\mathcal{U}}$ into $\mathbb{C}$ satisfying

$$(x_i)^2 - t((x_i))(x_i) + n((x_i))\mathbf{1} = 0$$

for all (x_i) in $(A_i)_{\mathcal{U}}$. Now, $(A_i)_{\mathcal{U}}$ is a quadratic algebra. Since $\varphi(A)$ is a $*$-invariant subalgebra of $(A_i)_{\mathcal{U}}$ with dimension ≥ 2, it follows from the second assertion in Lemma 3 that $\varphi(A)$ (and hence A) is quadratic.

Finally assume that Condition 3 is satisfied. As in the previous case, we are provided with an ultrafilter $\mathcal{U}$ on I such that the $*$-homomorphism $\varphi : x \to (\varphi_i(x))$ from A to $(A_i)_{\mathcal{U}}$ is injective. Put $\lambda := \lim_{\mathcal{U}} \lambda_i$. Then we easily obtain

$$(A_i)_{\mathcal{U}} = (B_i^{(\lambda_i)})_{\mathcal{U}} = (B_i^{(\lambda)})_{\mathcal{U}} = ((B_i)_{\mathcal{U}})^{(\lambda)} \ .$$

If $\lambda = \frac{1}{2}$, then $(A_i)_{\mathcal{U}}$ (and hence A) is commutative. Otherwise, we are in the situation which follows: $\frac{1}{2} < \lambda \leq 1$, $D := (B_i)_{\mathcal{U}}$ is a C^*-algebra, and, through the (automatically isometric) injective $*$-homomorphism φ, A can be seen as a closed $*$-invariant subalgebra of $D^{(\lambda)}$. Then, since the product $x \circ y$ of D can be determined from the one xy of $D^{(\lambda)}$ by means of the equality

$$x \circ y = \frac{1}{2\lambda - 1}(\lambda xy + (\lambda - 1)yx) \ ,$$

there exists a closed $*$-invariant subalgebra of D (say B) satisfying $A = B^{(\lambda)}$. Now the proof is concluded by showing that the C^*-algebra B is prime. Let P and Q be ideals of B with $P \circ Q = 0$. Then, clearly, P and Q are ideals of A. On the other hand, since C^*-algebras are semiprime, $P \circ Q = 0$ implies $Q \circ P = 0$, and therefore $PQ \subseteq P \circ Q + Q \circ P = 0$. By the primeness of A, we have either $P = 0$ or $Q = 0$. ■

Quadratic prime non-commutative JB^*-algebras have been precisely described in [9, Section 3]. According to that description, they are in fact Type I non-commutative JBW^*-factors. For commutative prime JB^*-algebras, the reader is referred to [5; Theorem 2.3].

Recall that a *W^*-algebra* is a C^*-algebra which is a dual Banach space, and that a *W^*-factor* is a prime W^*-algebra. The next result follows directly from Theorem 4.

COROLLARY 5 ([1], [3]).- *Non-commutative JBW^*-factors are either commutative, quadratic, or of the form $B^{(\lambda)}$ for some W^*-factor B and some real number λ with $\frac{1}{2} < \lambda \leq 1$.*

For (commutative) JBW^*-factors, the reader is referred to [5; Proposition 1.1].

As we have pointed out earlier, for non-commutative JBW^*-factors of Type I, the W^*-factor B arising in the above Corollary is equal to the algebra $BL(H)$ of all bounded linear operators on some complex Hilbert space H. This result follows from Corollary 5 and the fact that the algebras of the form $BL(H)$, with H a complex Hilbert space, are the unique W^*-factors of Type I [6; Proposition 7.5.2]. Concerning (commutative) JBW^*-factors of Type I, we can invoke the categorical correspondence between JBW-algebras and JBW^*-algebras [4], to reformulate the classification of JBW-factors of Type I [6; Corollary 5.3.7, and Theorems 5.3.8, 6.1.8, and 7.5.11] in the terms given by the next proposition. We recall that a *conjugation* (respectively, *anticonjugation*) σ on a complex Hilbert space H is a conjugate-linear isometry from H to H satisfying $\sigma^2 = \mathbf{1}$ (respectively, $\sigma^2 = -\mathbf{1}$).

PROPOSITION 6.- *The JBW^*-factors of Type I are the following:*

1. *The exceptional JB^*-algebra $H_3(\mathbb{O}_{\mathbb{C}})$.*
2. *The prime quadratic JB^*-algebras.*
3. *The JB^*-algebras of the form $B^{(1/2)}$, where $B = BL(H)$ for some complex Hilbert space H.*
4. *The JB^*-algebras of the form $\{x \in BL(H) : \sigma^{-1}x^*\sigma = x\}$, where H is a complex Hilbert space, and σ is either a conjugation or an anticonjugation on H.*

A normed algebra A is called *topologically simple* if $A^2 \neq 0$ and the unique closed ideals of A are $\{0\}$ and A.

COROLLARY 7.- *Topologically simple non-commutative JB^*-algebras are either commutative, quadratic, or of the form $B^{(\lambda)}$ for some topologically simple C^*-algebra B and some real number λ with $\frac{1}{2} < \lambda \leq 1$.*

Proof.- Since topologically simple normed algebras are prime, Theorem 4 applies. But, if A is a non-commutative JB^*-algebra of the form $B^{(\lambda)}$, for some C^*-algebra B and some $\frac{1}{2} < \lambda \leq 1$, then every ideal of B is an ideal of A, and hence B is topologically simple whenever A is so. ■

We note that every quadratic prime JB^*-algebra is algebraically (hence topologically) simple. For topologically simple (commutative) JB^*-algebras, the reader is referred to [5; Corollary 3.1].

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